Midterm exam Linear Algebra II

Friday 13/03/2026, 14:00–16:00

The midterm exam consists of three questions. Please write clearly and **motivate** all your answers. Good luck!

1 $(18 = 2+2+2+2+3+(2+2)+3$ pts) **subspace and linear operator**

Let $\mathbb{S}^2$ denote the set of real symmetric 2×2 matrices.

(a) Show that $\mathbb{S}^2$ is a subspace of $\mathbb{R}^{2\times 2}$.

(b) Show that

$$\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)$$

is a basis of $\mathbb{S}^2$.

(c) Let $A \in \mathbb{R}^{2\times 2}$ and define $L_A : \mathbb{S}^2 \to \mathbb{S}^2$ as

$$L_A(P) = P - A^\top P A.$$

Prove that L_A is a linear operator.

(d) Give (with proof!) an example of a matrix $A \in \mathbb{R}^{2\times 2}$ such that $\ker L_A \neq \{0\}$.

(e) We call a matrix $A \in \mathbb{R}^{2\times 2}$ *Schur* if $\lim_{k\to\infty} A^k = 0$. Assume that $A \in \mathbb{R}^{2\times 2}$ is Schur. Prove that $\ker L_A = \{0\}$.

(f) Now, let $Q \in \mathbb{S}^2$ and consider the *Lyapunov equation*

$$L_A(P) = Q \tag{1}$$

in the variable $P \in \mathbb{S}^2$. Assume that A is Schur. Prove that

$$P := \sum_{k=0}^{\infty} (A^\top)^k Q A^k$$

satisfies (1). Also prove that this P is the *unique* real symmetric solution to (1).

(g) For a general $A \in \mathbb{R}^{2\times 2}$, compute the matrix representation of L_A with respect to the basis in (b).

Hint: Write this matrix representation in terms of the entries $a_{11}, a_{12}, a_{21}, a_{22}$ of A.

2 $(9 = 3 + (1 + 2) + 3$ pts) **kernel, image, dimension theorem**

In this exercise V is an F-vector space with $\dim V < \infty$, and $T: V \to V$ is F-linear.

(a) Give (with proof!) an example of this situation where moreover the conditions $\ker T = T(V)$ and $\dim V \neq 0$ hold.

(b) Show that in general if $\ker T = T(V)$ then $\dim V$ is an even integer, and $T \circ T$ is the zero-map.

(c) Give (again, with proof!) an example showing that the converse of (b) is false. In other words: an example with the linear map T not the zero-map and $T \circ T$ equal to the zero map, and $\dim V$ even, but $\ker T \neq T(V)$.

3 $(9 = 4 + 2 + 3$ pts) **inner product space, norm**

In this exercise we consider in the $\mathbb{R}$-vector space $\mathbb{R}^\infty$ (consisting of all real sequences $(a_n)_{n\geq 1}$ with term by term addition and scalar multiplication), the subspace

$$V := \{(a_n)_{n\geq 1} \in \mathbb{R}^\infty \ : \ 3a_{n+3} = a_{n+2} + a_{n+1} + a_n \ \text{ for all } \ n \geq 1\}.$$

Given $c \in \mathbb{R}$, we moreover define $\langle \cdot, \cdot \rangle_c : V \times V \to \mathbb{R}$ by

$$\langle (a_n)_{n\geq 1}, (b_n)_{n\geq 1} \rangle_c := a_2 b_2 + a_3 b_3 + 9c a_4 b_4.$$

(a) Show that $\langle \cdot, \cdot \rangle_c$ defines an inner product on V if and only if $c > 0$.

(b) Take $c = 1$. Show that the constant sequence $(1)_{n\geq 1}$ is in V, and find its norm w.r.t. the norm defined by $\langle \cdot, \cdot \rangle_1$.

(c) Again take $c = 1$. Give (with proof!) a basis for the subspace of V consisting of all $(a_n)_{n\geq 1} \in V$ satisfying $(a_n)_{n\geq 1} \perp (1)_{n\geq 1}$.

4 pts free